

COHOMOLOGICAL CHARACTERIZATION OF  $T$ -LAU PRODUCT ALGEBRAS

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ABSTRACT. Let  $A$  and  $B$  be Banach algebras and let  $T$  be an algebra homomorphism from  $B$  into  $A$ . The Cartesian product space  $A \times B$  by  $T$ -Lau product and  $\ell^1$ -norm becomes a Banach algebra  $A \times_T B$ . We investigate the notions such as injectivity, projectivity and flatness for the Banach algebra  $A \times_T B$ . We also characterize Hochschild cohomology for the Banach algebra  $A \times_T B$ .

## 1. INTRODUCTION AND PRELIMINARIES

Suppose that  $A$  and  $B$  are Banach algebras and  $T : B \rightarrow A$  is an algebra homomorphism. Then we consider the Cartesian product space  $A \times B$  with the following multiplication

$$(a, b) \times_T (c, d) = (ac + T(b)c + aT(d), bd) \quad ((a, b), (c, d) \in A \times B),$$

which is denoted by  $A \times_T B$ . Let  $\|T\| \leq 1$ . Then we consider  $A \times_T B$  with the following norm

$$\|(a, b)\| = \|a\| + \|b\| \quad ((a, b) \in A \times_T B).$$

We note that  $A \times_T B$  is a Banach algebra with this norm and it is called  $T$ -Lau product algebras

Whenever the Banach algebra  $A$  is commutative, Bhatt and Dabshi [1] have investigated the properties of the Banach algebra  $A \times_T B$ , such as Gelfand space, Arens regularity and amenability.

Whenever  $A$  is unital with unit element  $e$  and  $\varphi : B \rightarrow \mathbb{C}$  is a character on  $B$ , assume  $T : B \rightarrow A$  is defined by  $T(b) = \phi(b)e$ . In this case the multiplication  $\times_T$  corresponds with the product studied by Lau [10]. Lau product was extended by Sangani Monfared for the general case and many basic properties of this product are studied in [13].

In the definition of  $T$ -Lau product, we can replace condition  $\|T\| \leq 1$  with a bounded algebra homomorphism  $T$ , because if we consider the following norms

$$\|a\|_T = \|T\| \|a\| \quad (a \in A)$$

$$\|b\|_T = \|T\| \|b\| \quad (b \in B)$$

$$\|(a, b)\|_T = \|a\|_T + \|b\|_T \quad (a, b) \in A \times_T B,$$

then all these norms are equivalent with the original norms. Clearly all results of this paper hold when we consider these equivalent norms.

The authors in [12] for every Banach algebras  $A$  and  $B$  and for an algebra homomorphism  $T : B \rightarrow A$  with  $\|T\| \leq 1$  have investigated some homological properties of  $T$ -Lau product algebra  $A \times_T B$  such as

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approximate amenability, pseudo amenability,  $\phi$ -pseudo amenability,  $\phi$ -biflatness and  $\phi$ -biprojectivity and have presented the characterization of the double centralizer algebra of  $A \times_T B$ .

Following [12], in this paper we studied the homological notions such as injectivity, projectivity and flatness for the Banach algebra  $A \times_T B$ . We also characterize the Hochschild cohomology for the Banach algebra  $A \times_T B$ .

## 2. INJECTIVITY, FLATNESS AND PROJECTIVITY

Let  $A$  be a Banach algebra. In this paper, the category of Banach left  $A$ -modules and Banach right  $A$ -modules is denoted by  $A\text{-}\mathbf{mod}$  and  $\mathbf{mod}\text{-}A$ , respectively. We denote by  $B(E, F)$  the Banach space of all bounded operators from  $E$  into  $F$ . In the category of  $A\text{-mod}$ , we denote the space of bounded morphisms from  $E$  into  $F$  by  ${}_A B(E, F)$ . A function  $S \in B(E, F)$  is called admissible if there exists  $S' \in B(F, E)$  such that  $S \circ S' \circ S = S$ .

A. Ya. Helemskii introduced the concepts of injectivity and flatness for Banach algebras [5] and these concepts have been investigated for different classes of Banach modules in [3, 4, 11, 14].

**Definition 2.1.** A Banach left  $A$ -module  $K$  is called projective if for every admissible epimorphism  $S : E \rightarrow F$  in  $A\text{-}\mathbf{mod}$ , the induced map  $S_A : {}_A B(K, E) \rightarrow {}_A B(K, F)$  defined by

$$S_A(R_A) = R_A \circ S \quad (R_A \in {}_A B(K, E))$$

is surjective.

**Definition 2.2.** A Banach left  $A$ -module  $K$  is called injective if for every admissible monomorphism  $S : E \rightarrow F$  in  $A\text{-}\mathbf{mod}$ , the induced map  $S_A : {}_A B(F, K) \rightarrow {}_A B(E, K)$  defined by

$$S_A(R_A) = R_A \circ S \quad (R_A \in {}_A B(F, K))$$

is surjective.

**Definition 2.3.** A Banach left  $A$ -module  $K$  is called flat if the dual module  $K^*$  in  $\mathbf{mod}\text{-}A$  is injective with the action defined by

$$(f \cdot a)(x) = f(a \cdot x),$$

where  $a \in A$ ,  $x \in K$  and  $f \in K^*$ .

Let  $A, B$ , and  $C$  be Banach algebras and let  $T : B \rightarrow A$  be an algebra homomorphism with  $\|T\| \leq 1$ . We note that if  $A$  is a Banach left  $C$ -module, then  $A \times_T B$  is a Banach left  $C$ -module via the following action

$$c \cdot (a, b) = (c \cdot a + c \cdot T(b), 0) \quad ((a, b) \in A \times_T B, c \in C).$$

Similarly if  $B$  is a Banach left  $C$ -module, then  $A \times_T B$  is a Banach left  $C$ -module via the following action

$$c \cdot (a, b) = (-T(c \cdot b), c \cdot b).$$

**Theorem 2.4.** Suppose that  $A$  and  $B$  are Banach algebras and  $T : B \rightarrow A$  is an algebra homomorphism with  $\|T\| \leq 1$ . Suppose that  $C$  is Banach algebra and  $A \times_T B$  is injective as  $C$ -module. Then  $A$  and  $B$  are injective as  $C$ -module.

*Proof.* Let  $A$  be a Banach left  $\mathcal{C}$ -module and let  $F, K \in \mathcal{C}\text{-}\mathbf{mod}$ . Suppose that  $S \in {}_{\mathcal{C}}B(F, K)$  is admissible and monomorphism. We will show that the induced map  $S_A : {}_{\mathcal{C}}B(K, A) \rightarrow {}_{\mathcal{C}}B(F, A)$  is onto.

We conclude that  $A \times_T B \in \mathcal{C}\text{-}\mathbf{mod}$  via the following action

$$c \cdot (a, b) = (c \cdot a + c \cdot T(b), 0).$$

Since  $A \times_T B$  is injective, the induced map  $S_{A \times_T B} : {}_{\mathcal{C}}B(K, A \times_T B) \rightarrow {}_{\mathcal{C}}B(F, A \times_T B)$  is onto. Let  $\lambda \in {}_{\mathcal{C}}B(F, A)$  and  $f \in F$ . We define  $\tilde{\lambda} : F \rightarrow A \times_T B$  by  $\tilde{\lambda}(f) = (\lambda(f), 0)$ . Hence we have  $\tilde{\lambda} \in {}_{\mathcal{C}}B(F, A \times_T B)$ . Since  $S_{A \times_T B} : {}_{\mathcal{C}}B(K, A \times_T B) \rightarrow {}_{\mathcal{C}}B(F, A \times_T B)$  is onto, there exists  $R_{A \times_T B} : K \rightarrow A \times_T B$  such that  $R_{A \times_T B}(T(f)) = \tilde{\lambda}(f) = (\lambda(f), 0)$ . We define  $R_A : K \rightarrow A$  by  $R_A = P_A \circ R_{A \times_T B}$ , where  $P_A : A \times_T B \rightarrow A$  is defined by  $p_A(a, b) = a$ . Clearly  $R_A \in {}_{\mathcal{C}}B(K, A)$  and  $R_A \circ T' = \lambda$ . So  $A$  is injective.

For injectivity of  $B$ , suppose that  $B$  is a Banach left  $\mathcal{C}$ -module,  $F$  and  $K \in \mathcal{C}\text{-}\mathbf{mod}$  and  $S \in {}_{\mathcal{C}}B(F, K)$  is admissible and monomorphism. We must show that the induced map  $S_B : {}_{\mathcal{C}}B(K, B) \rightarrow {}_{\mathcal{C}}B(F, B)$  is onto. We have  $A \times_T B \in \mathcal{C}\text{-}\mathbf{mod}$  with the following actions

$$c \cdot (a, b) = (-T(c \cdot b), c \cdot b).$$

Since  $A \times_T B$  is injective, the induced map  $S_{A \times_T B} : {}_{\mathcal{C}}B(K, A \times_T B) \rightarrow {}_{\mathcal{C}}B(F, A \times_T B)$  is onto. Let  $\mu \in {}_{\mathcal{C}}B(F, B)$  and  $f \in F$ . We define  $\tilde{\mu} : F \rightarrow A \times_T B$  by  $\tilde{\mu}(f) = (0, \mu(f))$ . Then we have  $\tilde{\mu} \in {}_{\mathcal{C}}B(F, A \times_T B)$ . Since  $S_{A \times_T B} : {}_{\mathcal{C}}B(K, A \times_T B) \rightarrow {}_{\mathcal{C}}B(F, A \times_T B)$  is onto, there exists  $R_{A \times_T B} : K \rightarrow A \times_T B$  such that  $R_{A \times_T B}(T(f)) = \tilde{\mu}(f) = (0, \mu(f))$ . We define  $R_A : K \rightarrow A$  by  $R_A = P_A \circ R_{A \times_T B}$ . Clearly  $R_A \in {}_{\mathcal{C}}B(K, B)$  and  $R_A \circ S = \mu$ . Hence  $B$  is injective. This completes the proof.  $\square$

Let  $A, B$  and  $\mathcal{C}$  be Banach algebras and let  $T : B \rightarrow A$  be an algebra homomorphism with  $\|T\| \leq 1$ . We note that if  $A \times_T B$  is the Banach left  $\mathcal{C}$ -module, then  $A$  and  $B$  can be the Banach left  $\mathcal{C}$ -modules with the following actions

$$c \cdot a = c \cdot (a, 0) \quad \text{and} \quad c \cdot b = c \cdot (0, b),$$

where  $c \in \mathcal{C}, a \in A$  and  $b \in B$ .

**Theorem 2.5.** *Suppose that  $A$  and  $B$  are Banach algebras and  $T : B \rightarrow A$  is an algebra homomorphism with  $\|T\| \leq 1$ . Suppose that  $\mathcal{C}$  is a Banach algebra and  $A$  and  $B$  are injective as  $\mathcal{C}\text{-}\mathbf{mod}$ . Then  $A \times_T B$  is injective as  $\mathcal{C}\text{-}\mathbf{mod}$ .*

*Proof.* Let  $A \times_T B$  be a Banach left  $\mathcal{C}$ -module and  $F, K \in \mathcal{C}\text{-}\mathbf{mod}$ . Let  $S \in {}_{\mathcal{C}}B(F, K)$  such that  $S$  is admissible and monomorphism. We must show that the induced map  $S_{A \times_T B} : {}_{\mathcal{C}}B(K, A \times_T B) \rightarrow {}_{\mathcal{C}}B(F, A \times_T B)$  is onto. We have  $A, B \in \mathcal{C}\text{-}\mathbf{mod}$  with the following actions

$$c \cdot a = c \cdot (a, 0) \quad \text{and} \quad c \cdot b = c \cdot (0, b),$$

where  $c \in \mathcal{C}, a \in A$  and  $b \in B$ . Since  $A$  and  $B$  are injective, the induced maps  $S_A : {}_{\mathcal{C}}B(K, A) \rightarrow {}_{\mathcal{C}}B(F, A)$  and  $S_B : {}_{\mathcal{C}}B(K, B) \rightarrow {}_{\mathcal{C}}B(F, B)$  are onto. Suppose that  $\lambda \in {}_{\mathcal{C}}B(F, A \times_T B)$  and  $(a, b) \in A \times_T B$  such that  $\lambda(f) = (a, b)$  for  $f \in F$ .

We define  $\tilde{\lambda} : F \rightarrow A$  by  $\tilde{\lambda}(f) = a + T(b)$  and  $\tilde{\mu} : F \rightarrow B$  by  $\tilde{\mu}(f) = b$ . Hence we have  $\tilde{\lambda} \in {}_{\mathcal{C}}B(F, A)$  and  $\tilde{\mu} \in {}_{\mathcal{C}}B(F, B)$ . Since  $S_A : {}_{\mathcal{C}}B(K, A) \rightarrow {}_{\mathcal{C}}B(F, A)$  and  $S_B : {}_{\mathcal{C}}B(K, B) \rightarrow {}_{\mathcal{C}}B(F, B)$  are onto, there exist  $R_A : K \rightarrow A$  and  $R_B : K \rightarrow B$  such that  $R_A \circ S(f) = \tilde{\lambda}(f) = a + T(b)$  and  $R_B \circ S(f) = \tilde{\mu}(f) = b$ .

We define  $R_{A \times_T B} : K \rightarrow A \times_T B$  by  $R_{A \times_T B} = q_A \circ R_A + \eta_B \circ R_B$ , where  $\eta_B : B \rightarrow A \times_T B$  such that  $\eta_B(b) = (-T(b), b)$ . Clearly  $R_{A \times_T B} \in {}_{\mathcal{C}}B(K, A \times_T B)$  and  $R_{A \times_T B} \circ S = \lambda$ . So  $A \times_T B$  is injective.  $\square$

Let  $A, B$  and  $\mathcal{C}$  be Banach algebras. We note that if  $A^* \times B^*$  is a Banach left  $\mathcal{C}$ -module, then  $A^*$  and  $B^*$  can be consider as Banach left  $\mathcal{C}$ -modules via the following actions

$$c \cdot a^* = c \cdot (a^*, 0) \quad \text{and} \quad c \cdot b^* = c \cdot (0, b^*),$$

where  $c \in \mathcal{C}, a^* \in A^*$  and  $b^* \in B^*$ .

**Theorem 2.6.** *Suppose that  $A$  and  $B$  are Banach algebras and  $T : B \rightarrow A$  is an algebra homomorphism with  $\|T\| \leq 1$ . Suppose that  $\mathcal{C}$  is a Banach algebra. Then  $A \times_T B$  is flat as **mod- $\mathcal{C}$**  if and only if  $A$  and  $B$  are flat as **mod- $\mathcal{C}$** .*

*Proof.* Let  $A \times_T B$  be flat as **mod- $\mathcal{C}$** . With a simple argument we can show that  $(A \times_T B)^* \cong A^* \times B^*$ . Hence by similar argument as in Theorem 2.4, one can show that  $A$  and  $B$  are flat Banach algebras as **mod- $\mathcal{C}$**

Conversely, let  $A$  and  $B$  be flat Banach algebras as **mod- $\mathcal{C}$** , let  $F, K \in \mathcal{C}\text{-mod}$  and let  $S \in {}_{\mathcal{C}}B(F, K)$  such that  $S$  is admissible and monomorphism. Then we show that the induced map  $S_{A^* \times B^*} : {}_{\mathcal{C}}B(K, A^* \times B^*) \rightarrow {}_{\mathcal{C}}B(F, A^* \times B^*)$  is onto. We have  $A^*, B^* \in \mathcal{C}\text{-mod}$  with the following actions

$$c \cdot a^* = c \cdot (a^*, 0) \quad \text{and} \quad c \cdot b^* = c \cdot (0, b^*),$$

where  $c \in \mathcal{C}, a^* \in A^*$  and  $b^* \in B^*$ .

Since  $A^*$  and  $B^*$  are injective as left  $\mathcal{C}$ -module, the induced maps  $S_{A^*} : {}_{\mathcal{C}}B(K, A^*) \rightarrow {}_{\mathcal{C}}B(F, A^*)$  and  $S_{B^*} : {}_{\mathcal{C}}B(K, B^*) \rightarrow {}_{\mathcal{C}}B(F, B^*)$  are onto. Suppose that  $\lambda^* \in {}_{\mathcal{C}}B(F, A^* \times B^*)$  and  $(a^*, b^*) \in A^* \times B^*$  such that  $\lambda^*(f) = (a^*, b^*)$  for  $f \in F$ .

We define  $\widetilde{\lambda}^* : F \rightarrow A^*$  by  $\widetilde{\lambda}^*(f) = a^*$  and  $\widetilde{\mu}^* : F \rightarrow B^*$  by  $\widetilde{\mu}^*(f) = b^*$ . Hence we have  $\widetilde{\lambda}^* \in {}_{\mathcal{C}}B(F, A^*)$  and  $\widetilde{\mu}^* \in {}_{\mathcal{C}}B(F, B^*)$ . Since  $S_{A^*} : {}_{\mathcal{C}}B(K, A^*) \rightarrow {}_{\mathcal{C}}B(F, A^*)$  and  $S_{B^*} : {}_{\mathcal{C}}B(K, B^*) \rightarrow {}_{\mathcal{C}}B(F, B^*)$  are onto, there exist  $R_{A^*} : K \rightarrow A^*$  and  $R_{B^*} : K \rightarrow B^*$  such that  $R_{A^*} \circ S(f) = \widetilde{\lambda}^*(f) = a^*$  and  $R_{B^*} \circ S(f) = \widetilde{\mu}^*(f) = b^*$ .

We define  $R_{A^* \times B^*} : K \rightarrow A^* \times B^*$  by  $R_{A^* \times B^*} = q_{A^*} \circ R_{A^*} + q_{B^*} \circ R_{B^*}$ , where  $q_{A^*} : A^* \rightarrow A^* \times B^*$  and  $q_{B^*} : B^* \rightarrow A^* \times B^*$  are defined by  $q_{A^*}(a^*) = (a^*, 0)$  and  $q_{B^*}(b^*) = (0, b^*)$ , respectively. Clearly  $R_{A^* \times B^*} \in {}_{\mathcal{C}}B(K, A^* \times B^*)$  and  $R_{A^* \times B^*} \circ S = \lambda^*$ . So  $A^* \times B^*$  is injective as left  $\mathcal{C}$ -module. This completes the proof.  $\square$

Let  $A, B$  and  $\mathcal{C}$  be Banach algebras and let  $T : B \rightarrow A$  be an algebra homomorphism with  $\|T\| \leq 1$ . If  $A \times_T B$  is a Banach left  $\mathcal{C}$ -module, as we have seen before,  $A$  and  $B$  are Banach left  $\mathcal{C}$ -modules.

**Theorem 2.7.** *Suppose that  $A$  and  $B$  are Banach algebras and  $T : B \rightarrow A$  is an algebra homomorphism with  $\|T\| \leq 1$ . Suppose that  $\mathcal{C}$  is a Banach algebra. Then  $A \times_T B$  is projective as  **$\mathcal{C}$ -mod** if and only if  $A$  and  $B$  are projective as  **$\mathcal{C}$ -mod**.*

*Proof.* Let  $A \times_T B$  be projective as Banach left  $\mathcal{C}$ -module and  $F, K \in \mathcal{C}\text{-}\mathbf{mod}$ . Let  $S \in {}_{\mathcal{C}}B(K, F)$  be admissible and epimorphism. We show that the induced map  $S_{A \times_T B} : {}_{\mathcal{C}}B(A \times_T B, K) \rightarrow {}_{\mathcal{C}}B(A \times_T B, F)$  is onto.

Since  $A, B \in \mathcal{C}\text{-}\mathbf{mod}$  are projective, the induced map  $S_A : {}_{\mathcal{C}}B(A, K) \rightarrow {}_{\mathcal{C}}B(A, F)$  and  $S_B : {}_{\mathcal{C}}B(B, K) \rightarrow {}_{\mathcal{C}}B(B, F)$  are onto. Let  $\lambda \in {}_{\mathcal{C}}B(A \times_T B, F)$  and  $f_1, f_2 \in F$  such that  $\lambda(a, 0) = f_1, \lambda(0, b) = f_2$  for  $(a, 0), (0, b) \in A \times_T B$ . We define  $\tilde{\lambda} : A \rightarrow F$  by  $\tilde{\lambda}(a) = \lambda(a, 0) = f_1$  and  $\tilde{\mu} : B \rightarrow F$  by  $\tilde{\mu}(b) = \lambda(0, b) = f_2$ . Hence we have  $\tilde{\lambda} \in {}_{\mathcal{C}}B(A, F)$  and  $\tilde{\mu} \in {}_{\mathcal{C}}B(B, F)$ . Since  $S_A : {}_{\mathcal{C}}B(A, K) \rightarrow {}_{\mathcal{C}}B(A, F)$  and  $S_B : {}_{\mathcal{C}}B(B, K) \rightarrow {}_{\mathcal{C}}B(B, F)$  are onto, there exist  $R_A \in {}_{\mathcal{C}}B(A, K)$  and  $R_B \in {}_{\mathcal{C}}B(B, K)$  such that  $S \circ R_A(a) = \tilde{\lambda}(a) = f_1$  and  $S \circ R_B(b) = \tilde{\mu}(b) = f_2$ . We define  $R_{A \times_T B} : A \times_T B \rightarrow K$  by  $R_{A \times_T B} = R_A \circ P_A + R_B \circ P_B$ . Clearly  $R_{A \times_T B} \in {}_{\mathcal{C}}B(A \times_T B, K)$  and  $T' \circ R_{A \times_T B} = \lambda$ . Hence  $A \times_T B$  is projective as left  $\mathcal{C}$ -module.

Conversely, let  $A$  be a Banach left  $\mathcal{C}$ -module and  $F, K \in \mathcal{C}\text{-}\mathbf{mod}$  and let  $S \in {}_{\mathcal{C}}B(K, F)$  such that  $S$  be admissible and epimorphism. We show that the induced map  $S_A : {}_{\mathcal{C}}B(A, K) \rightarrow {}_{\mathcal{C}}B(A, F)$  is onto. We have  $A \times_T B \in \mathcal{C}\text{-}\mathbf{mod}$  with the following action

$$c \cdot (a, b) = (c \cdot a + c \cdot T(b), 0).$$

Since  $A \times_T B$  is projective, the induced map  $S_{A \times_T B} : {}_{\mathcal{C}}B(A \times_T B, K) \rightarrow {}_{\mathcal{C}}B(A \times_T B, F)$  is onto. Let  $\lambda \in {}_{\mathcal{C}}B(A, F)$  and  $f \in F$  such that  $\lambda(a) = f$  for  $a \in A$ . We define  $\tilde{\lambda} : A \times_T B \rightarrow F$  by  $\tilde{\lambda}(a, b) = \lambda(a + T(b))$ . Hence we have  $\tilde{\lambda} \in {}_{\mathcal{C}}B(A \times_T B, F)$ . Since  $S_{A \times_T B} : {}_{\mathcal{C}}B(A \times_T B, K) \rightarrow {}_{\mathcal{C}}B(A \times_T B, F)$  is onto, there exists  $R_{A \times_T B} \in {}_{\mathcal{C}}B(A \times_T B, K)$  such that  $S \circ R_{A \times_T B} = \tilde{\lambda}$ . We define  $R_A : A \rightarrow K$  by  $R_A = R_{A \times_T B} \circ q_A$ , where  $q_A : A \rightarrow A \times_T B$  is defined by  $q_A(a) = (a, 0)$ . Clearly  $R_A \in {}_{\mathcal{C}}B(A, K)$  and  $S \circ R_A = \lambda$ . Hence  $A$  is projective as left  $\mathcal{C}$ -module.

For the proof of projectivity of  $B$ , let  $B$  be a Banach left  $\mathcal{C}$ -module and  $F, K \in \mathcal{C}\text{-}\mathbf{mod}$  and let  $S \in {}_{\mathcal{C}}B(K, F)$  such that  $S$  is admissible and epimorphism. We show that the induced map  $S_B : {}_{\mathcal{C}}B(B, K) \rightarrow {}_{\mathcal{C}}B(B, F)$  is onto. We have  $A \times_T B \in \mathcal{C}\text{-}\mathbf{mod}$  with the following action

$$c \cdot (a, b) = (-T(c \cdot b), c \cdot b).$$

Since  $A \times_T B$  is projective as left  $\mathcal{C}$ -module, the induced map  $S_{A \times_T B} : {}_{\mathcal{C}}B(A \times_T B, K) \rightarrow {}_{\mathcal{C}}B(A \times_T B, F)$  is onto. Let  $\lambda \in {}_{\mathcal{C}}B(B, F)$ . We define  $\tilde{\lambda} : A \times_T B \rightarrow F$  by  $\tilde{\lambda}(a, b) = \lambda(b)$  for  $(a, b) \in A \times_T B$ . Hence  $\tilde{\lambda} \in {}_{\mathcal{C}}B(A \times_T B, F)$ . Since  $S_{A \times_T B} : {}_{\mathcal{C}}B(A \times_T B, K) \rightarrow {}_{\mathcal{C}}B(A \times_T B, F)$  is onto, there exists  $R_{A \times_T B} \in {}_{\mathcal{C}}B(A \times_T B, K)$  such that  $S \circ R_{A \times_T B}(a, b) = \tilde{\lambda}(a, b) = \lambda(b)$ . We define  $R_B : B \rightarrow K$  by  $R_B = R_{A \times_T B} \circ q_B$ , where  $q_B : B \rightarrow A \times_T B$  is defined by  $q_B(b) = (0, b)$  for  $b \in B$ . Clearly  $R_B \in {}_{\mathcal{C}}B(B, K)$  and  $S \circ R_B = \lambda$ . Hence  $B$  is projective as left  $\mathcal{C}$ -module.  $\square$

### 3. HOCHSCHILD COHOMOLOGY FOR THE BANACH ALGEBRA $A \times_T B$

The concept of Hochschild cohomology for Banach algebras has been studied by Kamowitz[9], Johnson[7, 8] and others. Recall that let  $A$  be a Banach algebra and let  $X$  be a Banach  $A$ -bimodule. We denote the space of bounded  $n$ -linear maps from  $A$  into  $X$  by  $\mathcal{C}^n(A, X)$ . For  $T \in \mathcal{C}^n(A, X)$  we define the map

$\delta^n : \mathcal{C}^n(A, X) \rightarrow \mathcal{C}^{n+1}(A, X)$  by

$$\begin{aligned} (\delta^n T)(a_1, \dots, a_{n+1}) &= a_1 \cdot T(a_2, \dots, a_{n+1}) \\ &\quad + \sum_{i=1}^n (-1)^i T(a_1, \dots, a_i a_{i+1}, \dots, a_{n+1}) \\ &= (-1)^{n+1} T(a_1, \dots, a_n) \cdot a_{n+1}. \end{aligned}$$

$T$  is called an  $n$ -cocycle if  $\delta^n T = 0$  and it is called  $n$ -coboundary if there exists  $S \in \mathcal{C}^{n-1}(A, X)$  such that  $T = \delta^{n-1} S$ . We denote the linear space of all  $n$ -cocycles by  $\mathcal{Z}^n(A, X)$  and the linear space of all  $n$ -coboundaries by  $\mathcal{B}^n(A, X)$ . Clearly  $\mathcal{Z}^n(A, X)$  includes  $\mathcal{B}^n(A, X)$ . We also recall that the  $n$ -th Hochschild cohomology group  $\mathcal{H}^n(A, X)$  is defined by the following quotient,

$$\mathcal{H}^n(A, X) = \frac{\mathcal{Z}^n(A, X)}{\mathcal{B}^n(A, X)},$$

for more details, see [8]. We remark that a left (right) Banach  $A$ -module  $X$  is called left (right) essential if the linear span of  $A \cdot X = \{a \cdot x : a \in A, x \in X\}$  ( $X \cdot A = \{x \cdot a : x \in X, a \in A\}$ ) is dense in  $X$ . A Banach  $A$ -module  $X$  is called essential, if it is left and right essential.

Let  $A$  and  $B$  be Banach algebras and  $T : B \rightarrow A$  be an algebra homomorphism with  $\|T\| \leq 1$ . Let  $E$  be a Banach  $A$ -bimodule. Then  $E$  is also a Banach  $B$ -bimodule and a Banach  $A \times_T B$ -bimodule with the following actions, respectively

$$b \cdot x = T(b) \cdot x \quad \text{and} \quad x \cdot b = x \cdot T(b),$$

where  $b \in B, x \in E$  and

$$(a, b) \cdot x = T(b) \cdot x \quad \text{and} \quad x \cdot (a, b) = x \cdot T(b),$$

where  $(a, b) \in A \times_T B, x \in E$ .

**Lemma 3.1.** *Let  $E$  be an essential Banach  $A \times_T B$ -bimodule. Then  $E$  is an essential Banach  $A$ -bimodule and an essential Banach  $B$ -bimodule.*

**Theorem 3.2.** *Let  $A$  and  $B$  be Banach algebras with bounded approximate identity and let  $T : B \rightarrow A$  be an algebra homomorphism with  $\|T\| \leq 1$ . Let  $E$  be an essential  $A \times_T B$ -bimodule. Then*

$$\mathcal{H}^1(A \times_T B, E^*) \simeq \mathcal{H}^1(A, E^*) \times \mathcal{H}^1(B, E^*),$$

where  $\simeq$  denotes the vector space isomorphism.

*Proof.* By [2, Theorem 2.9.53] we have  $\mathcal{H}^1(A \times_T B, E^*) \simeq \mathcal{H}^1(\mathcal{M}(A \times_T B), E^*)$ , where  $\mathcal{M}(A \times_T B)$  denotes the double centralizer algebra of  $A \times_T B$ . But from [12, Theorem 4.3] we have

$$\mathcal{M}(A \times_T B) \cong \mathcal{M}(A) \times \mathcal{M}(B),$$

where  $\cong$  denotes the algebra isomorphism. this implies that

$$\mathcal{H}^1(\mathcal{M}(A \times_T B), E^*) \simeq \mathcal{H}^1(\mathcal{M}(A) \times \mathcal{M}(B), E^*),$$

where  $\mathcal{M}(A)$  and  $\mathcal{M}(B)$  denote the double centralizer algebra of  $A$  and  $B$ , respectively.

Hence we have

$$\mathcal{H}^1(A \times_T B, E^*) \simeq \mathcal{H}^1(\mathcal{M}(A) \times \mathcal{M}(B), E^*),$$

Using [6, Theorem 4] we obtain  $\mathcal{H}^1(\mathcal{M}(A) \times \mathcal{M}(B), E^*) \simeq \mathcal{H}^1(\mathcal{M}(A), E^*) \times \mathcal{H}^1(\mathcal{M}(B), E^*)$ , thus we have

$$\mathcal{H}^1(A \times_T B, E^*) \simeq \mathcal{H}^1(\mathcal{M}(A), E^*) \times \mathcal{H}^1(\mathcal{M}(B), E^*).$$

Since  $E$  is an essential Banach  $A$ -bimodule and an essential Banach  $B$ -bimodule, we have

$$\mathcal{H}^1(\mathcal{M}(A), E^*) \simeq \mathcal{H}^1(A, E^*)$$

and

$$\mathcal{H}^1(\mathcal{M}(B), E^*) \simeq \mathcal{H}^1(B, E^*).$$

This completes the proof.  $\square$

We can extend the previous theorem for the  $n$ -th Hochschild cohomology for the Banach algebra  $A \times_T B$ .

**Corollary 3.3.** *Let  $A$  and  $B$  be Banach algebras with bounded approximate identity and let  $T : B \rightarrow A$  be an algebra homomorphism with  $\|T\| \leq 1$ . Let  $E$  be an essential  $A \times_T B$ -bimodule. Then*

$$\mathcal{H}^n(A \times_T B, E^*) \simeq \mathcal{H}^n(A, E^*) \times \mathcal{H}^n(B, E^*)$$

for every  $n \geq 1$ .

*Proof.* By [12, Lemma 3.1]  $A$  and  $B$  have bounded approximate identities if and only if  $A \times_T B$  has a bounded approximate identity. Using [2, Theorem 2.9.54] one can show that if  $A \times_T B$  has a bounded approximate identity, then for every essential  $A \times_T B$ -bimodule  $E$ , we have  $\mathcal{H}^n(A \times_T B, E^*) \simeq \mathcal{H}^n(M(A \times_T B), E^*)$ . In [12, Theorem 4.3] the authors showed that  $M(A \times_T B) \cong M(A) \times M(B)$ . On the other hand Hochschild [6, Theorem 4] showed that  $\mathcal{H}^n(M(A) \times M(B), E^*) \simeq \mathcal{H}^n(M(A), E^*) \times \mathcal{H}^n(M(B), E^*)$ . Hence we have  $\mathcal{H}^n(A \times_T B, E^*) \simeq \mathcal{H}^n(A, E^*) \times \mathcal{H}^n(B, E^*)$  where  $n \geq 1$ .  $\square$

Note that Bhatt and Dabshi in [1] showed that  $A \times_T B$  is amenable if and only if  $A$  and  $B$  are amenable. In the essential case this is an immediate corollary of Theorem 3.2.

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